ELSEVIER Journal of Computational and Applied Mathematics 58 (1995) 171–181

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

The stability of the θ-methods in the numerical solution of delay differential equations with several delay terms

Tian Hongjiong*, Kuang Jiaoxun

Department of Mathematics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, People's Republic of China

Received 9 March 1993; revised 26 October 1993

Abstract

This paper deals with the stability analysis of numerical methods for the solution of delay differential equations (DDEs). We focus on the stability behaviour of the θ-methods in the solution of the following linear test equation with m delay terms:

$$y'(t) = ay(t) + \sum_{j=1}^{m} b_j y(t - \tau_j), \quad t \geqslant 0,$$

$$y(t) = \phi(t), \quad t \leqslant 0,$$

where a, b_j $(j = 1, 2, \ldots, m) \in \mathbb{C}, \tau_m \geqslant \tau_{m-1} \geqslant \cdots \geqslant \tau_1 > 0$, and $\phi(t)$ is continuous and complex valued. For $m = 2$, it is shown that the linear θ-method and the new θ-method are GP_2-stable if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$ and that the one-leg θ-method is GP_2-stable if and only if $\theta = 1$. In addition, for $m > 2$, we investigate the stability properties of the θ-methods with respect to the linear test equation and arrive at the same results.

Keywords: $(\delta_1, \delta_2, \ldots, \delta_m)$-stable; GP_m-stable; Schur polynomial

1. Introduction

This paper deals with the numerical solution of the initial-value problem

$$y'(t) = f(t, y(t), y(\alpha_1[t]), \ldots, y(\alpha_m[t])), \quad t \geqslant 0, \tag{1.1a}$$

$$y(t) = \phi(t), \quad t \leqslant 0. \tag{1.1b}$$

* Corresponding author.

SSDI 0377-0427(93)E0269-R

Here f, ϕ, α_j denote given functions with $\alpha_j[t] \leqslant t$ $(j = 1, 2, \ldots, m)$, whereas $y(t)$ is unknown (for $t > 0$).

The stability of numerical methods for (1.1) has been considered in [1–4, 8–10, 13] based on the following linear delay differential equation:

$$y'(t) = ay(t) + by(t - \tau), \quad t \geqslant 0, \tag{1.2a}$$

$$y(t) = g(t), \quad t \leqslant 0, \tag{1.2b}$$

where $a, b \in \mathbb{C}$, $\tau > 0$, and $g(t)$ is a specified initial function. It is known that [1, 2], if $g(t)$ is continuous and if

$$|b| < -\operatorname{Re}(a), \tag{1.3}$$

then the solution $y(t)$ of (1.2) tends to zero as $t \to \infty$.

In [2], the author introduced the concepts of P- and GP-stability.

Definition 1. A numerical method for DDEs (1.2) is called *P-stable* if, for all coefficients a, b satisfying (1.3), the numerical solution y_k of (1.2) at the mesh points $t_k = kh$, $k \geqslant 0$, satisfies

$$y_k \to 0, \quad \text{as } k \to \infty,$$

for every stepsize h such that $h = \tau/n$, where n is a positive integer.

Definition 2. A numerical method for DDEs (1.2) is called *GP-stable* if, under condition (1.3),

$$y_k \to 0, \quad \text{as } k \to \infty,$$

for every stepsize $h > 0$.

Many authors have dealt with the P-stability and GP-stability of various kinds of numerical methods [1, 3, 9, 10]. The numerical stabilities of the θ-methods have been carefully studied in [3, 9, 10]. In [10], the authors have shown that the linear θ-method is GP-stable with respect to (1.2) if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$ and the one-leg θ-method is GP-stable if and only if $\theta = 1$.

In the following sections, we will focus on the stability behaviour of the θ-methods with respect to the following linear test problems with m delay terms:

$$y'(t) = ay(t) + \sum_{j=1}^{m} b_j y(t - \tau_j), \quad t \geqslant 0, \tag{1.4a}$$

$$y(t) = \phi(t), \quad t \leqslant 0, \tag{1.4b}$$

where a, b_j $(j = 1, 2, \ldots, m) \in \mathbb{C}$, $\tau_m \geqslant \tau_{m-1} \geqslant \cdots \geqslant \tau_1 > 0$, and $\phi(t)$ is a specified initial function. For $m = 2$, it is shown that the linear θ-method and the new θ-method for DDEs (1.4) are GP_2-stable if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$, and the one-leg θ-method is GP_2-stable with respect to (1.4) if and only if $\theta = 1$. Further, for $m > 2$, we investigate the stability properties of the θ-methods for the linear test equation (1.4). It is proven that the linear θ-method and new θ-method are GP_m-stable if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$, and the one-leg θ-method is GP_m-stable if and only if $\theta = 1$.

2. Asymptotic stability of differential equations with several delay terms

First of all, we consider the linear test equation (1.4).

If one seeks an exponential solution of (1.4) in the form

$$y(t) = \mathrm{e}^{st}c, \tag{2.1}$$

where s, c are constants [5], then (1.4) has a nontrivial solution if and only if

$$s - a - b_1\mathrm{e}^{-s\tau_1} - \cdots - b_m\mathrm{e}^{-s\tau_m} = 0. \tag{2.2}$$

Definition 3. The DDE (1.4) is called *asymptotically stable* if, for any continuous initial function $\phi(t)$ and for any delay terms τ_j, $j = 1, 2, \ldots, m$, the solution

$$y(t) \to 0, \quad \text{as } t \to \infty.$$

The following theorem plays a key role in finding the sufficient condition for the asymptotic stability of (1.4).

Theorem 4 (Driver [5]). *For any delay terms $\tau_j > 0$, $j = 1, 2, \ldots, m$, all continuous solutions of* (1.4) *approach zero as $t \to \infty$ if and only if all the roots of* (2.2) *have negative real parts.*

The following theorem gives the sufficient condition for the asymptotic stability of (1.4).

Theorem 5. *If a, b_j, $j = 1, 2, \ldots, m$, satisfy*

$$\mathrm{Re}(a) < 0, \tag{2.3a}$$

$$\sum_{j=1}^{m} |b_j| < -\mathrm{Re}(a), \tag{2.3b}$$

then for any $\tau_j > 0$, $j = 1, 2, \ldots, m$, all the roots of Eq. (2.2) *have negative real parts.*

Proof. Let $s = x + \mathrm{i}\,y$ be the solution of (2.2). The characteristic equation of (1.4) becomes

$$x + \mathrm{i}\,y = a + \sum_{j=1}^{m} b_j \mathrm{e}^{-x\tau_j - \mathrm{i}\,y\tau_j}.$$

Let $b_j = |b_j|\mathrm{e}^{\mathrm{i}\phi_j}$.

If $x \geqslant 0$, then

$$\begin{aligned} x - \mathrm{Re}(a) &= \sum_{j=1}^{m} |b_j|\mathrm{e}^{-x\tau_j}\cos(-y\tau_j + \phi_j) \\ &\leqslant \sum_{j=1}^{m} |b_j|. \end{aligned}$$

Furthermore,

$$-\operatorname{Re}(a) \leqslant \sum_{j=1}^{m} |b_j|;$$

this contradicts (2.3b) and completes the proof of this theorem. □

Corollary 6. *If* a, b_j, $j = 1, 2, \ldots, m$, *satisfy*

$$\operatorname{Re}(a) < 0, \qquad \sum_{j=1}^{m} |b_j| < -\operatorname{Re}(a),$$

then for any $\tau_j > 0$, $j = 1, 2, \ldots, m$, *the DDE* (1.4) *is asymptotically stable.*

3. Numerical stability of the linear θ-method

The following method, called the linear θ-method, was considered, e.g., in [3, 8–10, 13], for DDEs (1.1) when $\alpha_1[t] = \alpha_2[t] = \cdots = \alpha_m[t]$:

$$\begin{aligned} y_{s+1} = y_s &+ \theta h f((s+1)h, y_{s+1}, y^h(\alpha_1[(s+1)h]), \ldots, y^h(\alpha_m[(s+1)h])) \\ &+ (1-\theta) h f(sh, y_s, y^h(\alpha_1[sh]), \ldots, y^h(\alpha_m[sh])) \end{aligned} \tag{3.1}$$

for $s = 0, 1, \ldots$. Here θ is a parameter, with $0 \leqslant \theta \leqslant 1$, specifying the method. Further $y_0 = \phi(0)$, $y^h(t) = \phi(t)$ (for $t \leqslant 0$), and $y^h(t)$ with $t > 0$ is defined by piecewise linear interpolation, i.e.

$$y^h(t) = (t - kh)h^{-1} y_{k+1} + ((k+1)h - t)h^{-1} y_k \quad \text{for } kh < t \leqslant (k+1)h; \quad k = 0, 1, 2, \ldots. \tag{3.2}$$

In this section, we shall investigate the numerical stability of the linear θ-method in the numerical solution of the test problems (1.4).

Applying (3.1) in combination with (3.2) to (1.4), we arrive at the following recurrence relation:

$$\begin{aligned} y_{s+1} = (1 - \theta\bar{a})^{-1} \Bigg\{ &[1 + (1-\theta)\bar{a}] y_s + \theta \sum_{j=1}^{m} \bar{b}_j [y_{s+2-n_j}\delta_j + (1-\delta_j) y_{s+1-n_j}] \\ &+ (1-\theta) \sum_{j=1}^{m} \bar{b}_j [y_{s+1-n_j}\delta_j + (1-\delta_j) y_{s-n_j}] \Bigg\}. \end{aligned} \tag{3.3}$$

Here $s \geqslant n_m \geqslant n_{m-1} \geqslant \cdots \geqslant n_1, n_j, j = 1, 2, \ldots, m$, are the smallest integers with $\tau_j h^{-1} \leqslant n_j$, $\delta_j = n_j - \tau_j h^{-1}$, $\delta_j \in [0, 1)$, $\bar{a} = ha$, $\bar{b}_j = hb_j$, $j = 1, 2, \ldots, m$. The values y_j, $j = 0, 1, \ldots, n_m$, which are needed to start the numerical process (3.3), are obtained from the initial function $\phi(t)$ and from application of (3.3) (with $s = 0, 1, \ldots, n_m - 1$).

For given $\delta_1, \delta_2, \ldots, \delta_m \in [0, 1)$ and $(\bar{a}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m) \in \mathbb{C}^{m+1}$, we introduce some definitions of stability based on DDEs (1.4).

Definition 7. A numerical method for DDEs (1.4) is called $(\delta_1, \delta_2, \ldots, \delta_m)$-*stable* at $(\bar{a}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m)$ if any application of the method to (1.4) yields approximations y_s with $y_s \to 0$,

$s \to \infty$, whenever h, a, b_j, $j = 1, 2, \ldots, m$, are given with $\bar{a} = ha, \bar{b}_j = hb_j, j = 1, 2, \ldots, m$, $h = (n_j - \delta_j)^{-1}\tau_j, j = 1, 2, \ldots, m$, and integers $n_m \geqslant n_{m-1} \geqslant \cdots \geqslant n_1 \geqslant 1$.

The subset of $\mathbb{C}^{m+1}$ consisting of all $(\bar{a}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m)$ at which the method is $(\delta_1, \delta_2, \ldots, \delta_m)$-stable is called the $(\delta_1, \delta_2, \ldots, \delta_m)$-*stability region*. For the linear θ-method we denote this set by $S_{\theta, \delta_1, \delta_2, \ldots, \delta_m}$. The stability region S_θ of the θ-method is defined by

$$S_\theta = \bigcap_{\substack{0 \leqslant \delta_j < 1 \\ j = 1, 2, \ldots, m}} S_{\theta, \delta_1, \delta_2, \ldots, \delta_m}. \tag{3.4}$$

In view of the conditions of Corollary 6, we introduce the set

$$H = \{(\bar{a}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m) \in \mathbb{C}^{m+1}: \mathrm{Re}(\bar{a}) < 0, \textstyle\sum_{j=1}^{m} |\bar{b}_j| < -\mathrm{Re}(\bar{a})\}.$$

Definition 8. The linear θ-method for DDEs (1.4) is called GP_m-*stable* if and only if

$$H \subset S_\theta. \tag{3.5}$$

Case I: $m = 2$.

To the recurrence relation (3.3) we adjoin the characteristic polynomial

$$P_{n_1, n_2}(z, \delta_1, \delta_2) = z^{n_2} Q_2(z) - z^{n_2 - n_1} Q_1(z, \delta_1) - P(z, \delta_2), \tag{3.6}$$

where

$$Q_1(z, \delta_1) = \bar{b}_1 (1 - \theta\bar{a})^{-1} [\theta z + (1 - \theta)][\delta_1 z + (1 - \delta_1)], \tag{3.7a}$$

$$Q_2(z) = z - (1 - \theta\bar{a})^{-1} [1 + (1 - \theta)\bar{a}], \tag{3.7b}$$

$$P(z, \delta_2) = \bar{b}_2 (1 - \theta\bar{a})^{-1} [\theta z + (1 - \theta)][\delta_2 z + (1 - \delta_2)]. \tag{3.7c}$$

By a well-known property of Schur polynomials [6], we obtain the following lemma.

Lemma 9. *Let* $\delta_1, \delta_2 \in [0, 1)$, $\bar{a}, \bar{b}_1, \bar{b}_2 \in \mathbb{C}$. *The numerical method* (3.1) *is* (δ_1, δ_2)-*stable if and only if the characteristic polynomial* (3.6) *is a Schur polynomial for* $n_2 \geqslant n_1 \geqslant 1$.

The following theorem gives a sufficient condition in order that $P_{n_1, n_2}(z, \delta_1, \delta_2)$ is a Schur polynomial for all $n_2 \geqslant n_1 \geqslant 1$.

Theorem 10. $P_{n_1, n_2}(z, \delta_1, \delta_2)$ *is a Schur polynomial for all* $n_2 \geqslant n_1 \geqslant 1$ *if*

$$|(1 - \theta\bar{a})^{-1} [1 + (1 - \theta)\bar{a}]| < 1, \tag{3.8}$$

$$|Q_1(z, \delta_1)| < |Q_2(z)|, \quad z \in c, \tag{3.9}$$

$$|P(z, \delta_2)| < |Q_2(z)| - |Q_1(z, \delta_1)|, \quad z \in c. \tag{3.10}$$

Here c denotes the unit circle in the complex plane.

Proof. From the statements (3.8) and (3.9), we may see that

$$Q(z,\delta_1) = z^{n_2}Q_2(z) - z^{n_2-n_1}Q_1(z,\delta_1)$$

is a Schur polynomial for all $n_2 \geqslant n_1 \geqslant 1$ by an application of Rouché's theorem [11].

Then $P_{n_1,n_2}(z,\delta_1,\delta_2)$ can be written in the form

$$P_{n_1,n_2}(z,\delta_1,\delta_2) = Q(z,\delta_1)[1 - \delta(z,\delta_1,\delta_2)], \tag{3.11}$$

where $\delta(z,\delta_1,\delta_2) = Q(z,\delta_1)^{-1}P(z,\delta_2)$ has a modulus $|\delta(z,\delta_1,\delta_2)| < 1$ for $|z| = 1$.

In the following, $\Delta_c \arg[f(z)]$ will stand for the increment of the argument of $f(z)$ when z runs through c. We have the following equality

$$\Delta_c \arg[P_{n_1,n_2}(z,\delta_1,\delta_2)] = \Delta_c \arg[Q(z,\delta_1)] + \Delta_c \arg[1 - \delta(z,\delta_1,\delta_2)]. \tag{3.12}$$

Since $|\delta(z,\delta_1,\delta_2)| < 1$ for $|z| = 1$, then we arrive at

$$\Delta_c \arg[1 - \delta(z,\delta_1,\delta_2)] = 0.$$

It follows that

$$\Delta_c \arg[P_{n_1,n_2}(z,\delta_1,\delta_2)] = \Delta_c \arg[Q(z,\delta_1)].$$

It can be proven that $P_{n_1,n_2}(z,\delta_1,\delta_2)$ is a Schur polynomial for all $n_2 \geqslant n_1 \geqslant 1$ by the principle of the argument.

This completes the proof. □

When dealing with the above condition (3.8), it is convenient to use the notation

$$P = (1 - 2\theta)^{-1} \quad \text{for } \theta \neq \tfrac{1}{2},$$

and to introduce the generalized disk D_θ, defined by

$$D_\theta = \{z \in \mathbb{C} : |z + p| < p\} \qquad \text{if } 0 \leqslant \theta < \tfrac{1}{2},$$

$$D_\theta = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\} \qquad \text{if } \theta = \tfrac{1}{2},$$

$$D_\theta = \{z \in \mathbb{C} : |z + p| > -p\} \quad \text{if } \tfrac{1}{2} < \theta \leqslant 1.$$

Let ∂D_θ denote the boundary of D_θ.

Since the function $w = (z - 1)/(1 - \theta + \theta z)$ maps the unit disk in the z-plane onto the generalized disk D_θ in the w-plane, it can be shown that the statements (3.9) and (3.10) are equivalent to (3.13) and (3.14), respectively:

$$|\bar{b}_1|\,|w\delta_1(1 - \theta w)^{-1} + 1| < |w - \bar{a}|, \quad w \in \partial D_\theta, \tag{3.13}$$

$$|\bar{b}_2|\,|w\delta_2(1 - \theta w)^{-1} + 1| < |w - \bar{a}| - |\bar{b}_1|\,|1 + w\delta_1(1 - \theta w)^{-1}|, \quad w \in \partial D_\theta. \tag{3.14}$$

Then we obtain the main theorem of this section.

Theorem 11. *Let $0 \leqslant \theta \leqslant 1$. Then the linear θ-method is* GP_2*-stable if and only if $\tfrac{1}{2} \leqslant \theta \leqslant 1$.*

Proof. (i) $0 \leqslant \theta < \frac{1}{2}$. From the definition of GP_2-stability, we can see that GP_2-stability implies GP-stability. From [10], we conclude that the linear θ-method is not GP_2-stable.

(ii) $\theta = \frac{1}{2}$. Let $\delta_j \in [0,1)$, $j = 1,2$, be arbitrary given and $(\bar{a}, \bar{b}_1, \bar{b}_2) \in H$. Since

$$|1 + \delta_j w(1 - \theta w)^{-1}| \leqslant 1, \quad w \in \partial D_\theta,\ j = 1,2,$$

we arrive at

$$|\bar{b}_1||1 + \delta_1 w(1 - \theta w)^{-1}| \leqslant |\bar{b}_1| < -\operatorname{Re}(\bar{a}) \leqslant |w - \bar{a}|, \quad w \in \partial D_\theta,$$

and

$$|\bar{b}_2||1 + \delta_2 w(1 - \theta w)^{-1}| + |\bar{b}_1||1 + \delta_1 w(1 - \theta w)^{-1}| \leqslant |\bar{b}_2| + |\bar{b}_1| < -\operatorname{Re}(\bar{a}) < |w - \bar{a}|,$$
$$w \in \partial D_\theta.$$

So the statements (3.9) and (3.10) are fulfilled.

Since $(\bar{a}, \bar{b}_1, \bar{b}_2) \in H$ implies $\operatorname{Re}(\bar{a}) < 0$,

$$|(1 - \bar{a}/2)^{-1}(1 + \bar{a}/2)| < 1,$$

then (3.8) holds.

It follows that the linear θ-method is GP_2-stable.

(iii) $\frac{1}{2} < \theta \leqslant 1$. The GP_2-stability of the linear θ-method can be obtained analogously.

So we complete the proof of this theorem. □

Case II: $m > 2$.

To the recurrence relation (3.3) we obtain the characteristic polynomial (3.15)

$$P_N(z;\Delta) = z^{n_m}\left[z - \frac{1 + (1-\theta)\bar{a}}{1 - \theta\bar{a}}\right] - \sum_{j=1}^{m} \frac{\bar{b}_j}{1 - \theta\bar{a}}[\theta z + (1-\theta)][\delta_j z + (1 - \delta_j)]z^{n_m - n_j}. \quad (3.15)$$

Analogously to Theorems 10 and 11, we may establish the following theorems.

Theorem 12. *$P_N(z;\Delta)$ is a Schur polynomial for $n_m \geqslant n_{m-1} \geqslant \cdots \geqslant n_1$ if and only if*

$$\left|\frac{1 + (1-\theta)\bar{a}}{1 - \theta\bar{a}}\right| < 1, \quad (3.16a)$$

$$\sum_{j=1}^{m}\left|\frac{\bar{b}_j}{1 - \theta\bar{a}}[\theta z + (1-\theta)][\delta_j z + (1-\delta_j)]\right| < \left|z - \frac{1 + (1-\theta)\bar{a}}{1 - \theta\bar{a}}\right|, \quad z \in c. \quad (3.16b)$$

Theorem 13. *For $m > 2$ and $0 \leqslant \frac{1}{2} \leqslant 1$, the linear θ-method is GP_m-stable if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$.*

4. Numerical stability of the one-leg θ-method

Consider the following one-leg θ-method :

$$y_{s+1} = y_s + hf((s+\theta)h, y^h[(s+\theta)h], y^h(\alpha_1[(s+\theta)h]), \dots, y^h(\alpha_m[(s+\theta)h])), \tag{4.1}$$

for $s = 0, 1, 2, \dots$; here θ is a parameter with $0 \leqslant \theta \leqslant 1$ specifying the method, $y_0 = \phi(0)$, $y^h(t) = \phi(t)$ for $t \leqslant 0$, and the definition of $y^h(t)$ for $t > 0$ is given by (3.2).

Substituting (4.1) and (3.2) into (1.4), we obtain

$$y_{s+1} = y_s + \left[\bar{a}(\theta y_{s+1} + (1-\theta) y_s) + \sum_{j=1}^{m} \bar{b}_j(\beta_j y_{s-r_j+1} + (1-\beta_j) y_{s-r_j}) \right]. \tag{4.2}$$

Here $s \geqslant r_m \geqslant r_{m-1} \geqslant \cdots \geqslant r_1$, $\bar{a} = ha$, $\bar{b}_j = hb_j, j = 1, 2, \dots, m, n_j$ are the smallest integers such that $\tau_j h^{-1} \leqslant n_j$, $\delta_j = n_j - \tau_j h^{-1} \in [0, 1)$,

$$\beta_j = \theta + \delta_j \text{ and } r_j = n_j \qquad \text{if } 0 \leqslant \delta_j < 1 - \theta,$$

$$\beta_j = \theta + \delta_j - 1 \text{ and } r_j = n_j - 1 \quad \text{if } 1 - \theta \leqslant \delta_j < 1,$$

for $j = 1, 2, \dots, m$.

Case I: $m = 2$.

We obtain the characteristic polynomial of (4.2):

$$\bar{P}_{r_1, r_2}(z, \delta_1, \delta_2) = z^{r_2} Q_2(z) - z^{r_2 - r_1} \bar{Q}_1(z, \delta_1) - \bar{P}(z, \delta_2), \tag{4.3}$$

where

$$\bar{Q}_1(z, \delta_1) = \bar{b}_1(1 - \theta\bar{a})^{-1}[\beta_1 z + (1 - \beta_1)], \tag{4.4a}$$

$$\bar{P}(z, \delta_2) = \bar{b}_2(1 - \theta\bar{a})^{-1}[\beta_2 z + (1 - \beta_2)]. \tag{4.4b}$$

Theorem 14. *Let $0 \leqslant \theta \leqslant 1$. The one-leg θ-method is* GP_2*-stable if and only if $\theta = 1$.*

Proof. (i) $0 \leqslant \theta < 1$. Since GP_2-stability implies GP-stability, it follows, from [10], that the one-leg θ-method is not GP_2-stable.

(ii) $\theta = 1$. The methods (3.1) and (4.1) coincide, from Theorem 11, such that the method is GP_2-stable. □

Case II: $m > 2$.

We can obtain the characteristic polynomial of (4.2):

$$\bar{P}_N(z; \Delta) = z^{r_m}\left[z - \frac{1 + (1-\theta)\bar{a}}{1 - \theta\bar{a}} \right] - \sum_{j=1}^{m} \frac{\bar{b}_j}{1 - \theta\bar{a}} [\beta_j z + (1 - \beta_j)] z^{r_m - r_j}. \tag{4.5}$$

Analogously to Theorem 14, we have the following theorem.

Theorem 15. *For $m > 2$ and $0 \leqslant \theta \leqslant 1$. The one-leg θ-method is* GP_m*-stable if and only if $\theta = 1$.*

5. Numerical stability of the new θ-method

Consider the following new θ-method [9]:

$$y_{s+1} = y_s + hf((s+\theta)h, \theta y_{s+1} + (1-\theta)y_s, \theta y^h(\alpha_1[(s+1)h]) + (1-\theta)y^h(\alpha_1[sh]),$$
$$\ldots, \theta y^h(\alpha_m[(s+1)h]) + (1-\theta)y^h(\alpha_m[sh])), \tag{5.1}$$

for $s = 0, 1, 2, \ldots$. Here θ is a parameter with $0 \leqslant \theta \leqslant 1$, specifying the method. Further $y_0 = \phi(0)$, $y^h(t) = \phi(t)$ for $t \leqslant 0$, and $y^h(t)$ with $t > 0$ is defined by (3.2).

Applying (5.1) and (3.2) to (1.4), we arrive at (3.3). So we have the following theorem.

Theorem 16. *Let $0 \leqslant \theta \leqslant 1$. Then the new θ-method is GP_m-stable if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$.*

6. Numerical experiments

In this section, we give a numerical illustration to the above theorems. Consider the following two problems with two delay terms and different stiffness features:

$$y'(t) = -50y(t) + 30y(t-1) + 10y(t-2) \ (t \geqslant 0), \qquad y(t) = \mathrm{e}^{-t} \ (t \leqslant 0), \tag{6.1}$$

$$y'(t) = -500y(t) + 300y(t-1) + 100y(t-2) \ (t \geqslant 0), \qquad y(t) = \mathrm{e}^{-t} \ (t \leqslant 0). \tag{6.2}$$

We shall compare various numerical processes for approximating the true solutions at $t = 10$, which are equal to $y(10) \simeq 0.259230656117$ (for (6.1)) and $y(10) \simeq 0.224079398319$ (for (6.2)).

Table 1
Numerical results with $\theta = \frac{1}{2}$ for problem (6.1)

	Method (3.1)		Method (4.1)	
	Constrained mesh	Free mesh	Constrained mesh	Free mesh
M	Error	Error	Error	Error
2	6.05E-01	9.91E-03	5.80E-01	1.04E-00
4	2.40E-02	3.52E-03	2.59E-02	1.32E-00
6	8.64E-03	3.11E-03	7.70E-03	9.04E-01
10	1.38E-04	1.28E-03	1.86E-04	1.52E-01
20	8.79E-06	3.87E-04	8.98E-05	2.55E-04
50	1.37E-06	6.88E-05	1.43E-05	3.98E-05
100	3.11E-08	1.77E-06	3.55E-06	9.75E-06
200	4.58E-08	4.50E-06	8.59E-07	2.38E-06

We deal with two different meshes used in [14, 10]. The first mesh consists of all gridpoints $t_n = nh \in [0,10]$ with $h = 1/M$ and fixed even integer $M \geqslant 2$. The second mesh contains all integers j with $0 \leqslant j \leqslant 10$ and in each interval $(j-1, j)$ with $1 \leqslant j \leqslant 10$ it contains all points t satisfying

$$t = [k + j(j+1)/22]h \quad \text{for some integer } k.$$

Here $h = 1/M$ as above. The first mesh is called a constrained mesh and the second one a free mesh. Both meshes include the breaking points of the solution, i.e. the integers $0, 1, \ldots, 10$.

In Table 1 we have listed the absolute values of errors that are present at $t = 10$ in the numerical solution of problem (6.1) when $\theta = \frac{1}{2}$. The two columns in the table pertinent to method (4.1) are mainly for comparing them with the corresponding columns pertinent to method (3.1).

Table 2
Numerical results with $\theta = \frac{1}{2}$ for problem (6.2)

	Method (3.1)		Method (4.2)	
	Constrained mesh	Free mesh	Constrained mesh	Free mesh
M	Error	Error	Error	Error
2	4.68E-00	9.73E-01	4.54E-00	7.30E-00
4	3.82E-01	2.56E-01	3.75E-01	2.97E + 01
6	7.13E-01	8.35E-02	7.09E-01	8.80E + 01
10	8.69E-02	1.11E-02	8.70E-02	1.40E + 01
20	1.05E-02	5.23E-04	1.04E-02	4.30E + 02
50	2.40E-07	3.01E-05	1.10E-05	1.30E-01
100	1.04E-08	1.09E-05	2.80E-06	8.61E-06
200	1.67E-08	3.40E-06	6.77E-07	2.11E-06

Table 3
Numerical results with $\theta = \frac{3}{4}$ for problem (6.2)

	Method (3.1)		Method (4.2)	
	Constrained mesh	Free mesh	Constrained mesh	Free mesh
M	Error	Error	Error	Error
2	2.81E-02	8.08E-03	3.30E-02	1.11E-01
4	1.06E-04	1.02E-03	1.26E-03	6.78E-02
6	3.10E-04	4.15E-04	2.90E-04	9.59E-03
10	8.83E-05	2.32E-04	1.25E-04	8.42E-04
20	4.47E-05	1.10E-04	8.22E-06	8.05E-05
50	1.79E-05	3.98E-05	9.52E-06	2.12E-06
100	8.99E-06	1.76E-05	6.90E-06	4.28E-06
200	4.51E-06	7.44E-06	3.98E-06	3.49E-06

In the table one sees a clear discrepancy between the error in method (4.1) using a free mesh and the errors in the other three numerical procedures considered. Note that for increasing M this phenomenon becomes less pronounced, in particular for $M \geqslant 20$.

Table 2 deals with the same methods as Table 1, but it concerns the numerical solution of (6.2). We note the surprising superiority of the linear method (3.1) over the one-leg method (4.1) on a free mesh with $M \simeq 20$.

Table 3 concerns the θ-methods with $\theta = \frac{3}{4}$ in the numerical solution of problem (6.2). We can see that method (4.1) on a free mesh produces less-accurate approximations than the other three numerical procedure, in particular for small M.

The above numerical experiments are in agreement with the theorems in this paper.

References

[1] A.N. Al-Mutib, Stability properties of numerical methods for solving delay differential equations, *J. Comput. Appl. Math.* **10** (1) (1984) 71–79.

[2] V.K. Barwell, Special stability problems for functional differential equations, *BIT* **15** (1975) 130–135.

[3] M. Calvo and T. Grande, On the asymptotic stability of the θ-methods for delay differential equations, *Numer. Math.* **54** (1988) 257–269.

[4] C.W. Cryer, Highly stable multistep methods for retarded differential equations, *SIAM J. Numer. Anal.* **11** (1974) 788–797.

[5] R.D. Driver, *Ordinary and Delay Differential Equations* (Springer, New York, 1977).

[6] R.T. Duffin, Algorithms for classical stability problems, *SIAM Rev.* **11** (1969) 196–213.

[7] J.K. Hale, *Theory of Functional Differential Equations* (Springer, New York, 1977).

[8] K.J. in 't Hout and M.N. Spijker, Stability analysis of numerical methods for delay differential equations, *Numer. Math.* **59** (1991) 807–814.

[9] K.J. in 't Hout and M.N. Spijker, The θ-methods in the numerical solution of delay differential equations, Rep. TW-89-03, Univ. Leiden, 1989.

[10] M.Z. Liu and M.N. Spijker, The stability of the θ-methods in the numerical solution of delay differential equations, *IMA J. Numer. Anal.* **10** (1) (1990) 31–48.

[11] M. Marden, *Geometry of Polynomials* (American Mathematical Society, Providence, RI, 1966).

[12] P.J. van der Houwen and B.P. Sommijer, Stability in linear multistep methods for pure delay equations, *J. Comput. Appl. Math.* **10** (1) (1984) 55–63.

[13] D.S. Watanabe and M.G. Roth, The stability of difference formulas for delay differential equations, *SIAM J. Numer. Anal.* **22** (1985) 132–145.

[14] M. Zennaro, P-stability properties of Runge–Kutta methods for delay differential equations, *Numer. Math.* **49** (1986) 305–318.